Theorem 1. If $A$ is square and has a row of zeros or a column of zeros, then $\det(A) =$

For the proof, consider a cofactor expansion along the row/column of zeros. What happens?

Theorem 2. Let $A$ be a square matrix. Then $\det(A^T) =$

For the proof: recall that to find the determinant of a matrix, we can do a cofactor expansion along any row or column of the matrix. How does this fact prove the theorem?

Theorem 3. Let $A$ be a square matrix.

(a) If one row or column of $A$ is multiplied by a scalar $c$ to produce $B$, then $\det(B) =$

(b) If two rows or columns of $A$ are interchanged to produce $B$, then $\det(B) =$

(c) If a multiple of one row (column) of $A$ is added to another row (column) of $A$ to produce $B$, then $\det(B) =$

For the proof of (a), consider how multiplying $A$ by a scalar $c$ changes the cofactor expansion you’ve chosen to use to get $\det(A)$.

For the proof of (b), find the following determinants (we’ll stick to the $2 \times 2$ and the $3 \times 3$ cases):

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

$$\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} =$$
\[ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \]

\[ \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = \]

For the proof of (c), we'll stick to the \(2 \times 2\) case. Find the determinant of the following matrix, where \(k \neq 0\):

\[ \begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = \]

**Example 1.** Do row reduction on the matrix \(A\) below to get to a matrix for which the determinant is easy to find (I’d suggest upper triangular). Keep track of your row operations as you go so that you can keep track of what happens with the determinant of the matrix. Check the answer you get for the determinant by taking the determinant of the unreduced matrix \(A\) using your calculator or whatever other means you choose.

(a)

\[ A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} \]
(b) \[
A = \begin{bmatrix}
2 & -8 & 6 & 8 \\
3 & -9 & 5 & 10 \\
-3 & 0 & 1 & -2 \\
1 & -4 & 0 & 6
\end{bmatrix}
\]

**Theorem 4.** Let $A$ be an $n \times n$ matrix with two proportional rows or two proportional columns. Then $\det(A) =$

For the proof, recall what it means for things to be proportional: if two rows (columns) are proportional, than one is a scalar multiple of the other. How does part (c) of the previous theorem allow us to prove this theorem quickly?